Intersecting subgroups of a free group

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Reference "On the intersection of subgroups of a free group" by Gábor Tardos in *Inventiones mathematicae*, 1992.

Definition The **rank** of a free group is the number of generators in a basis for that free group.

Conjecture (Hanna Neumann Conjecture (HNC)) *Given subgroups H and K of a free group F, where H has rank $n+1$ and K has rank $m+1$ $(n, m \geq 0)$, then the rank of the intersection $H \cap K$ has rank at most $nm+1$.*

Note that it is often convenient to talk about the reduced rank, $\tilde{r}(G) = \max(r(G) - 1, 0)$, which is the rank minus one, except that the reduced rank of the trivial group is 0. The Hanna Neumann Conjecture (now including the cases of H or K being trivial) then states $\tilde{r}(H \cap K) \leq \tilde{r}(H) \cdot \tilde{r}(K)$.

Conjecture (Strengthened Hanna Neumann Conjecture) *Given subgroups H and K of a free group F,*

$$\sum_{x \in X} \tilde{r}\left(x^{-1} H x \cap K\right) \leq \tilde{r}(H) \cdot \tilde{r}(K)$$

where $X \subset F$ is a set of doublecoset representatives for the doublecosets HxK.

Theorem (Hanna Neumann inequality) *Given subgroups H and K of a free group F,*

$$\tilde{r}(H \cap K) \leq 2\tilde{r}(H)\tilde{r}(K).$$

Proof I gave a talk on this last semester. □

Theorem (Tardos) *The Strengthened Hanna Neumann Conjecture holds when one of the groups has rank 2. I.e. if K has rank 2, then the rank of $K \cap H$ is at most the rank of H.*

Note: for all of the above, it is sufficient to show the case where F is the free group on two generators a, b.

Definition (Graphs) My graphs will be labeled and oriented (i.e.. every edge has a distinguished direction). So a graph Γ will consist of a vertex set V, an edge set E, a function $i : E \to V$ describing the initial vertex of each edge, a function $e \mapsto \bar{e} : E \to E$ taking each edge to the same edge except with the opposite orientation ($\bar{\bar{e}} = e$), and a function $l : E \to X$ taking edges to a label, which in our case will be one of $\{a, b, a^{-1}, b^{-1}\}$ ($l(\bar{e}) = l(e)^{-1}$). For every finite rank subgroup G of F, the free group on a, b, we can construct a finite graph Γ_G where $\pi_1(\Gamma_G) = G$ and there is an immersion of Γ_G into the bouquet of two circles which respects the labels of the graph.

Definition (The Pullback) We can construct the pullback of Γ_H, Γ_K (which I will write $\Gamma_H \times \Gamma_K$) as follows: let $V = V_H \times V_K$, and let $E = \{(e_1, e_2) : e_1 \in E_H, e_2 \in E_K, l(e_1) = l(e_2)\}$. Then, $i(e_1, e_2) = (i(e_1), i(e_2))$, $\overline{(e_1, e_2)} = (\bar{e_1}, \bar{e_2})$, $l(e_1, e_2) = l(e_1) = l(e_2)$. Note that there is a natural immersion of $\Gamma_H \times \Gamma_K$ into Γ_H or Γ_K by projection. The graph of $H \cap K$ will be one of the connected components of $\Gamma_H \times \Gamma_K$. The other components of the pullback correspond to the distinct conjugates of H intersected with K (as described in the strengthened version of the HNC).

Definition (Core) The core of a graph Γ is the maximal subgraph with no vertices of valence 1. Since we only care about the rank of the graph, i.e., how many loops it contains, we are only concerned about the cores of graphs. I will assume that all graphs are *normal*, that is have only vertices with valence strictly bigger than 1. Note that in the pullback construction above, we could first take core of Γ_H and Γ_K and that would not change the core of the result.

Definition (Branch vertices) Since our graphs will all immerse in the bouquet of two circles, the valences of vertices will be at most 4. Define $Y(\Gamma)$ to be the set of vertices of valence at least 3, and define $X(\Gamma)$ to be the set of vertices of valence 4. Define the *branching number* $b(\Gamma)$ to be $|Y(\Gamma)| + |X(\Gamma)|$. I.e. we count vertices of valence 3 plus twice vertices of valence 4. Observe that $\tilde{r}(G) = b(\Gamma_G)/2$ (assuming that we replace Γ_G with its core).

Notation If v is a vertex of Γ (i.e. $v \in V(\Gamma)$) and x is a word in F, then we say $vx \in \Gamma$ if the path starting at v along edges given by the letters of x. If such a path exists vx will be the vertex at the endpoint of the path. Note that if $vx \notin \Gamma$, we may extend Γ by adding at most one branch point.

Lemma 1 *Let Γ be a graph*[1]*, and $x = yzy^{-1}$ (when written as a reduced word, with $y, z \neq 1$) a conjugate element in F, then*

$$|\{v \in Y(\Gamma) : vx \in \Gamma\}| + \left|\{v \in Y(\Gamma) : vx^{-1} \in \Gamma\}\right| \leq b(\Gamma)$$

[1] Tardos says "normal graph" here, but his proof doesn't require the graph to be normal, and later he uses this lemma with graphs that are not normal.

Proof Subtract $|Y(\Gamma)|$ from both sides. The left side becomes the difference between the number of branch vertices v with both vx and vx^{-1} in Γ (call this $G(\Gamma)$), and the number of branch vertices with neither defined (call this $B(\Gamma)$). The right side simply becomes $|X(\Gamma)|$. We therefore want to show $|G(\Gamma)| \leq |B(\Gamma)| + |X(\Gamma)|$.

Assume without loss of generality that x, and therefore y, starts with a. Observe that z, z^{-1}, and y^{-1} all start with different first letters (say c, d, and e). This implies that for any $v \in G(\Gamma)$, $vyz, vyz^{-1} \in \Gamma$, so vy is also a branch point. If c, d, and e are all different than a, then there is a 1-1 map from $G(\Gamma)$ into the $B(\Gamma) \cup X(\Gamma)$ given by $v \mapsto vy$. The other cases are handled with an induction on the length of x (the basis is included in the case above): if $a = e$, i.e. y^{-1} starts with a, then y is a conjugate element with length less than x (can then show $|Y(\Gamma) \setminus B(\Gamma)| + |G(\Gamma)| \leq |\{v \in Y(\Gamma) : vy \in \Gamma\}| + |\{v \in Y(\Gamma) : vy^{-1} \in \Gamma\}|$). If $a = d$, so z^{-1} starts with a, then yz is a conjugate element shorter than x (and then we modify the graph). □

Lemma 2 *If Γ is a graph and $x \in F$ is a conjugate element, then*

$$|\{v \in Y(\Gamma) : vx \in \Gamma \textit{ or } vx^{-1} \in \Gamma\}| + |\{v \in V(\Gamma) : vx \in \Gamma \textit{ and } vx^{-1} \in \Gamma\}| \leq b(\Gamma)$$

Proof Note that this is the same as the previous lemma except the second set includes points which are not branches. All we do is extend the graph so that any non-branch point in the second set gets additional edges to bring the valence up to 3. Every time we do this, we increase both the size of the first set and $b(\Gamma)$ by 1. So if we apply the previous lemma to the extended graph, we get the desired inequality plus a constant added to both sides. □

This technique of adding new edges into the graph, applying the lemma and then accounting for the differences introduced will be used repeatedly.

Lemma 3 *If Γ is a graph and $x \in F$ is a conjugate element, then*

$$|\{v \in Y(\Gamma) : \exists n \neq 0 \textit{ s.t. } vx^n \in \Gamma\}| + |\{v \in V(\Gamma) : \exists m < 0 < n \textit{ s.t. } vx^m, vx^n \in \Gamma\}| \leq b(\Gamma)$$

Proof Note that if $x = yzy^{-1}$, then $x^n = yz^ny^{-1}$. So $vx^n \in \Gamma$ does not imply $vx \in \Gamma$. This proof follows the same pattern as the previous lemma. This time we extend the graph so that if $vx^n \in \Gamma$, then also $vx, vx^2, \ldots, vx^{n-1} \in \Gamma$. □

Main Idea We are going to show that if K has rank 2 then the rank of $H \cap K$ is at most the rank of H. Recall that the rank of a normal graph is directly proportional to its branch number. Our strategy is to count the possible branch points in the pullback graph $\Gamma_H \times \Gamma_K$ and show that there are corresponding branch points in Γ_H. Since we are dealing with the whole graph $\Gamma_H \times \Gamma_K$ instead of just the component $H \cap K$, we get the strengthened version of HNC for this particular case.

Figure 8 case Consider the case where Γ_K consists of two loops which share a single vertex, v. Since $\Gamma_K - \{v\}$ only has vertices of valence 2, all branch points in $\Gamma_H \times \Gamma_K$ are of the form $w' = (w, v)$ for some branch point $w \in \Gamma_H$. Further, the valence of w' is at most the valence of w. Therefore the branch number of the pullback is at most the branch number of Γ_H. (The only reason they are not equal is because the branch number of the pullback may decrease when we drop to its core.)

Spectacles case Consider Γ_K consisting of two loops connected by a path. Then Γ_K has two valence 3 vertices, s and t, a path y from s to t, a path z from t to itself, and a path from s to itself. Let $x = yzy^{-1}$. As before, any branch point in $\Gamma_H \times \Gamma_K$ must be the product of a branch point in Γ_H and a branch point in Γ_K (either s or t). Claims: If $v = (w, s)$ is a branch point, then $\exists n \neq 0$ s.t. $vx^n \in \Gamma_H \times \Gamma_K$. If (w, t) is a branch point, and $v = (w, t)y^{-1}$, then $\exists m < 0 < n$ s.t. $vx^m, vx^n \in \Gamma_H \times \Gamma_K$. Then simply apply the last lemma above, to get the result.

Theta case In this case Γ_K has two branch points, s and t, and three paths y, z_1, and z_2 from s to t. This time our conjugate element $x = yz_1^{-1}z_2y^{-1}$. Let V_s be the set of $v \in Y(\Gamma_H)$ called s-vertices such that $(s, v) \in \Gamma_H \times \Gamma_K$ (and similarly V_t, t-vertices). Note that it is possible for some s-vertices to also be t-vertices. Let $Y_s \subset Y(\Gamma_H)$ consist of those s-vertices v such that (v, s) is a branch point in $\Gamma_H \times \Gamma_K$ (similarly define Y_t). Observe that since we only are considering projections of the core of $\Gamma_H \times \Gamma_K$, $V_s \setminus Y_s$ consists of vertices v where two of vy, vz_1, and vz_2 are in Γ_H. Note that Y_s and Y_t are not necessarily disjoint either, but

$$b(\Gamma_H \times \Gamma_K) = |Y_s| + |Y_t|\,.$$

Now construct Γ'_H by building vy^{-1} from every t-vertex v in Γ_H, and then from every s-vertex $u \in \Gamma_H$ building both uz_1y^{-1} and uz_2y^{-1}. All that remains is to justify the following inequalities:

$$b(\Gamma'_H) - b(\Gamma_H) \leq |\{u \in V' \setminus V : ux \in \Gamma'_H \text{ and } ux^{-1} \in \Gamma'_H\}| + |\{u \in V_s \setminus Y_s : uy \in V_t\}|$$

(where V' is the set of vertices of Γ'_H)

$$|Y_t| \leq |\{u \in V : ux \in \Gamma'_H \text{ and } ux^{-1} \in \Gamma'_H\}|$$

$$|\{u \in V_s \setminus Y_s : uy \in V_t\}| + |Y_s| \leq |\{u \in Y(\Gamma'_H) : ux \in \Gamma'_H \text{ or } ux^{-1} \in \Gamma'_H\}|$$

Applying the second lemma to Γ'_H, gives:

$$|\{u \in Y(\Gamma'_H) : ux \in \Gamma'_H \text{ or } ux^{-1} \in \Gamma'_H\}| + |\{u \in V' : ux \in \Gamma'_H \text{ and } ux^{-1} \in \Gamma'_H\}| \leq b(\Gamma'_H)$$

Adding up all the inequalities along with $b(\Gamma_H \times \Gamma_K) = |Y_s| + |Y_t|$ gives the result $b(\Gamma_H \times \Gamma_K) \leq b(\Gamma_H)$ as desired.

Other Result Warren Dicks showed the Hanna Neumann Conjecture is equivalent to the Amalgamated Graph Conjecture, and used this to reprove the above result and another past result (Inv. math., 1994).

Other Result Then Dicks in 1996 proved the inequality when both subgroups have rank 3 (Inv. math, 1996).

Other Result Finally, in a preprint, Dicks along with E. Formanek showed $\tilde{r}(H \cap K) \leq \tilde{r}(H)\tilde{r}(K) + r_{-3}(H)r_{-3}(K)$ where $r_{-3}(H) = \max(r(H) - 3, 0)$. This proves that HNC holds when one subgroup has rank 3. See

`http://manwe.mat.uab.es/dicks/Rankthree.html`

(to appear). I believe this is the best known general bound.

Other Result The Hanna Neumann conjecture holds if H or K has a positive generating set (Bilal Khan, 2000).